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Observing gravitational wave bursts in pulsar timing measurements

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ABSTRACT
We propose a novel method for observing the gravitational wave signature of supermassive black hole (SMBH) mergers. This method is based on detection of a specific type of gravitational waves, namely gravitational wave burst with memory (BWM), using pulsar timing. We study the unique signature produced by BWM in anomalous pulsar timing residuals. We show that the present day pulsar timing precision allows one to detect BWM due to SMBH mergers from distances up to 1 Gpc (for case of equal mass $10^8\ M_\odot$ SMBH). Improvements in precision of pulsar timing together with the increase in number of observed pulsars should eventually lead to detection of a BWM signal due to SMBH merger, thereby making the proposed technique complementary to the capabilities of the planned LISA mission.

Key words: gravitational waves – galaxies: evolution – *(stars:)* pulsars: general – cosmology: miscellaneous

1 INTRODUCTION

The prospects of detecting gravitational waves (GWs) in the coming decade are looking ever more promising (Grishchuk et al. 2001; Cutler & Thorne 2002; Sathyaprakash & Schutz 2009). There is currently a considerable experimental effort to detect gravitational waves in a wide range of frequencies. At high frequencies $\nu \sim 10^1 - 10^3$ Hz the ground based laser interferometric detectors, such as the currently operational LIGO (LIGO 2009) and VIRGO (VIRGO 2009), especially in their "Advanced" configuration, should be able to observe GWs from a wide range of astrophysical sources. The planned space interferometer LISA (Baker et al. 2007) would be able to observe GWs in the region of frequencies $\nu \sim 10^{-4} - 10^{-1}$ Hz. At intermediate range of frequencies $\nu \sim 10^{-8} - 10^{-6}$ pulsar timing measurements are a very strong tool for observing the GW signature. Several pulsar timing array (PTA) projects, such as PPTA (Manchester 2007), EPTA (Janssen et al. 2008) and NANOGrav (Demorest et al. 2009) are collecting data and yielding an unprecedented and every increasing sensitivity in intermediate frequency range. Furthermore, the implementation of the planned Square Kilometer Array (SKA) (Cordes et al. 2004) radio telescope would further improve the sensitivity of pulsar timing measurements to gravitational waves. Finally, at the lowest frequencies $\nu \sim 10^{-18} - 10^{-16}$ Hz considerable efforts are being made to measure the imprints of relic gravitational waves in the temperature and polarization anisotropies of the cosmic microwave background (CMB) (Keating et al. 2006).

In what follows we shall consider pulsar timing, in particular timing of millisecond pulsar (MSPs), as a tool to study gravitational waves (see (Hobbs 2005; Jenet et al. 2005) for a recent overview). A GW passing between the pulsar and the observer on Earth will leave a modulating signature on the observed period of the pulsar. For this reason, the analysis of pulsar timing residuals gives a unique opportunity to observe GWs (Estabrook & Wahlquist 1975; Sazhin 1978; Detweiler 1979; Bertotti, Carr & Rees 1983; Kopeikin 1997). A particularly attractive method for detection is to cross-correlate the timing signal from different pulsars (Hellings & Downs 1983). This method led to the proposal for a PTA (Romani 1989; Foster & Backer 1990), which is aimed at observing a number of pulsars distributed on the sky at regular time intervals over a long time span (see Verbiest et al. (2009) for recent review).

Current pulsar timing observations are yet to detect gravitational waves. However, current upper bounds on the stochastic background of GWs $\Omega_{gw}h^2 < 2\cdot 10^{-8}$ (Jenet et al. 2006) (in terms of the ratio of energy density per unit logarithmic frequency

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interval to the cosmological critical density at frequency $\nu = 1/8$yrs) already place interesting limits on cosmic string models. Moreover, in the near future, it is likely that the GW background of coalescing extragalactic supermassive black hole binaries and possibly the background of relic gravitational waves will be discovered (Manchester 2007). Apart from the stochastic backgrounds, pulsar timing also has the potential to observe GWs from individual sources. In particular, there has been considerable interest in GWs emitted by massive black hole binaries systems (Jenet et al. 2004; Sesana et al. 2009).

In the present work we shall be interested with individual sources which emit gravitational waves of a particular class, namely gravitational wave bursts with memory (BWM) (Zel'Dovich & Polnarev 1974; Smarr 1977; Kovacs & Thorne 1978; Bontz & Price 1979; Braginsky & Grishchuk 1979; Braginsky & Thorne 1987), and analyze the prospects for their detection in pulsar timing measurements. In general, BWM are characterized by a rise in the gravitational wave field h_{ij}^{TT} from zero, followed by oscillatory behaviour for a few cycles and settling down at a final non-zero value Δh_{ij}^{TT} after a characteristic time δt known as the duration of the burst (Braginsky & Thorne 1987). The permanent change Δh_{ij}^{TT} is usually referred to as the burst's "memory". BWMs are produced in situations where there is a net change in the time derivatives of multipole moments characterizing the system, for example during flyby of massive bodies on hyperbolic trajectories or an asymmetric supernova explosion. A BWM where the net change of the moments is attributed to the gravitons released during the burst is known as the Christodoulou effect (Payne 1983; Christodoulou 1991; Thorne 1992; Blanchet & Damour 1992). Whatever the physical reason, the existence of the permanent offset Δh_{ij}^{TT} in BWMs makes them particularly interesting for pulsar timing studies. As we shall show below, the burst's memory leads to a characteristic signature in pulsar timing residuals which accumulates linearly with time.

The structure of the paper is as follows. We begin in Section 2 with a simple order-of-magnitude estimation of the potential sources of BWM, and compare them with the expected sensitivity of pulsar timing measurements. In this section, we study the typical amplitudes and possible event rates for potential signals. The simplistic analysis shows that, in the context of pulsar timing, the most observationally promising sources of BWM are extragalactic coalescing supermassive black holes binaries. In Section 3, we study the observational prospects for the detection of the BWM signal in significant detail. Using a simple analytical model for a BWM signal we calculate the pulsar timing residual. Following this, we evaluate the expected signal-to-noise ratio, and the expected number of BWM events that could be observed in a typical pulsar timing experiment. Finally, we conclude in Section 4 with an overview and discussion of the main results of this work.

2 POTENTIAL SOURCES OF GRAVITATIONAL WAVE BURSTS WITH MEMORY

2.1 Typical strength of BWM signal

In an astrophysical context, BWM typically occur in burst events which are accompanied by considerable amount of mass or radiation ejected in an asymmetric fashion. A simple formula for estimating the characteristic amplitude of BWM (Braginsky & Thorne 1987) reads:

$$h^{\rm mem} \sim \frac{r_g}{r}\left(\frac{v}{c}\right)^2, \tag{1}$$

where r_g and v are the Schwarzschild radius and velocity of aspherically ejected parts respectively, and r is the distance to the source of the burst. The effect is maximal when the mass is ejected at maximal speed, i.e c, corresponding to ejection of photons or gravitons. In the evaluations below we shall work in units $[G = c = 1]$.

An obvious source of GW BWM is a core-collapse supernova. In a typical scenario, the value of asymmetrically radiated energy is $\Delta E_{\rm rad} \leqslant 10^{-3} M_\odot$ (inferred from the velocities of neutron stars, see e.g. Nazin & Postnov (1997)), leading to an estimate $h_{\rm 1Mpc} \leqslant 10^{-22}$, for the typical BWM strain for a core-collapse supernova at a distance of 1 Mpc from the observer. In fact, these estimates are overoptimistic (see Ott (2009)). Taking into account the expected event rate of core-collapse supernovae of a few per century in a Milky Way type galaxy, one can conclude that the sensitivity of pulsar timing measurements will not allow the detection of such a signal (see Section 3.3 for a discussion of the signal-to-noise ratio).

Another interesting class of burst events are black hole (BH) mergers (Favata 2009). Depending on the angular momentum of the merging BHs, up to several percent of the total mass of the system can be radiated non-spherically during the burst (Reisswig et al. 2009). The masses of BHs range from stellar to several billion solar masses. However, from (1), we conclude that the most interesting candidates in the present context are the mergers of super massive black hole (SMBH) binaries, with typical masses $M_{\rm SMBH} \geqslant 10^8 M_\odot$. Assuming that 10% of the mass is radiated during the burst, for the BWM strain at a distance of 1 Gpc due to a SMBH merger with total mass $10^8 M_\odot$, we arrive at an estimate $h_{\rm 1Gpc} \simeq 10^{-15}$. As we shall show below, this strain is within the reach of pulsar timing measurements.

This estimate can be made more precise. The BWM amplitude from SMBH merger is given by Eq. (5) from Favata (2009):

$$h^{mem} = \frac{\eta M h}{384\pi r}\sin^2\theta(17+\cos^2\theta), \quad \text{where } h = \frac{16\pi}{\eta}\left(\frac{\Delta E_{\rm rad}}{M}\right). \tag{2}$$

In the above expression, $M = M_1 + M_2$ is the total mass of SMBH, $\eta = \frac{M_1+M_2}{M^2}$ and θ is the angle the binary angular momentum and the line of sight. This formula can be rewritten in the form:

$$h^{mem} = \frac{\Delta E_{\rm rad}}{24r}\sin^2\theta(17+\cos^2\theta). \tag{3}$$

Averaging over θ yields:

$$< h^{\rm mem} >= \frac{69}{8}\frac{\Delta E_{\rm rad}}{24r} \approx \frac{\Delta E_{\rm rad}}{3r}. \quad (4)$$

Estimates of $\Delta E_{\rm rad}$ can be obtained from calculations of Reisswig et al. (2009) which give $\Delta E_{\rm rad}$ from 3.6% to 10% M. In the present work, we shall use the mean value of $\Delta E_{\rm rad} = 7{\cdot}10^{-2}\ M$. For equal-mass SMBH system with $M_1 = M_2 = m = 10^8\ M_\odot$) and zero orbital eccentricity at a distance of $r = 1$ Gpc, we obtain:

$$h^{\rm mem} = 5 \cdot 10^{-16} \left(\frac{m}{10^8\ M_\odot}\right) \left(\frac{1\ {\rm Gpc}}{r}\right). \quad (5)$$

At this point it is instructive to compare the typical strain associated with the BWM signal from SMBH merger with the strain amplitude associated with the inspiral phase prior to the SMBH merger. The characteristic strain amplitude associated with the inspiral phase is given by (Sathyaprakash & Schutz 2009)

$$h^{\rm insp} \approx 10^{-15} \left(\frac{m}{10^8\ M_\odot}\right) \left(\frac{1\ {\rm Gpc}}{r}\right), \quad (6)$$

for the inspiral of SMBH of comparable masses. Although the characteristic strains associated with the BWM and the inspiral have comparable amplitudes, they lead to different timing residual signatures. In a nutshell, the qualitative difference in the residuals arises because the BWM signal is characterized by a permanent offset of GW field which leads to linear growth of the residuals with time, whereas the inspiral signal has a quasi-periodic time varying GW field and consequently does not lead to linear growth of the residuals (see Section 3).

In order to gain further insight, it is helpful to present a heuristic argument for concentrating our effort on the signature of BWM signal in pulsar timing residuals instead of the inspiral signal although both have the comparable characteristic strains. We refer the reader to Section 3.3 for a rigorous calculation, however for the purpose of order-of-magnitude estimation, it is reasonable to assume that the signal-to-noise ratios (SNR) associated with PTA measurement of BWM and inspiral signal are related as:

$$\frac{SNR_{\rm BWM}}{SNR_{\rm insp}} \sim \left(\frac{R_{\rm BWM}}{R_{\rm insp}}\right),$$

where $R_{\rm BWM}$, $R_{\rm insp}$ are timing residuals associated with the BWM and inspiral signal, respectively. The BWM residuals grow linearly with time $R_{\rm BWM} \sim h^{\rm mem} T_{\rm obs}$ (see Section 3), where $T_{\rm obs}$ is the total duration of observation. In the case of an inspiral signal due to its quasi-periodic nature the residuals oscillate with a characteristic amplitude $R_{\rm insp} \sim h^{\rm insp}/\omega_{\rm insp}$, where $\omega_{\rm insp} \sim \left[7.5 \cdot 10^3 \left(M/10^8 M_\odot\right) {\rm sec}\right]^{-1}$ is the characteristic frequency associated with the inspiral (which is the frequency at the Last Stable Circular Orbit). Thus, assuming $T_{\rm obs} = 10$ yrs, the expected ratio is:

$$\frac{SNR_{\rm BWM}}{SNR_{\rm insp}} \sim \frac{h^{\rm mem} T_{\rm obs}\omega_{\rm insp}}{h^{\rm insp}} \sim T_{\rm obs}\omega_{\rm insp} \sim 2 \cdot 10^4.$$

For this reason, in what follows, we shall ignore the possible contribution to the pulsar timing signal coming from inspiral of the SMBH binary.

2.2 Assessment of the event rate of SMBH mergers

The crude estimation above shows that the amplitude of BWM from SMBH mergers is within the expected reach of pulsar timing measurements. Thus, the prospect of detecting the BWM signal from SMBH binary mergers crucially depends on the expected event rate. A typical pulsar timing measurement lasts for a period of $T_{\rm obs} = 10$ yrs. For this reason, if the event rate in a volume of 1 Gpc^3 is larger or comparable to 0.1 events yr^{-1}, it is likely that a pulsar timing measurement would be able to detect such an event.

The event rate of SMBH mergers remains very uncertain. Various theoretical models predict rates that differ by 2-3 orders of magnitude (Baker et al. 2007; Volonteri 2006). The SMBH merger rate primarily depends on two factors both of which need further study. The first factor is the number of merging galaxies that contain SMBH in the mass range of our interest. The second factor is the fraction of galaxy mergers that leads to SMBH mergers. The galaxy merger rate at redshifts $z < 1$ is around one per year (see Baker et al. (2007) and references therein). Therefore, simply assuming that each of the merging galaxies contains an SMBH and that these SMBH coalesce in each galaxy merger, the SMBH merger rate should also be around one per year. Although very crude, these estimates indicate that the SMBH merger rate is in the right ball-park to be detected in pulsar timing measurements.

The event rate for SMBH mergers in the mass range $10^7\ M_\odot < M < 10^9\ M_\odot$ was calculated to be 0.4 events yr^{-1} in (Sesana et al. 2004). These mergers occur for redshifts $z < 4$, with maximum event rate at $z \sim 2$. In addition, their calculations show that at least over 20% of SMBH mergers have mass ratios larger than 0.2. A concordant number for the expected event rate, of 0.1 events yr^{-1} for SMBH merger at redshifts $z < 1$ with total mass $M \sim 10^8\ M_\odot$ comes from (Enoki et al. 2004). Moreover, as suggested by Fig. 6b in (Enoki et al. 2004), the event rate for SMBH mergers with masses $10^7\ M_\odot < M < 10^8\ M_\odot$ can be about 1 events yr^{-1}, albeit coming primarily from a larger redshift $z \sim 3$. Along with the theoretical estimates of the SMBH merger event rates, recent studies (Conselice,Yang & Bluck 2009) suggest $R_g \sim\ 10^{-3}$ events Gpc^{-3}yr^{-1} for the

rate of major galaxy mergers. Assuming that each major galaxy merger is associated with SMBH binary coalescence, both numerical estimations and observational data allows us to conclude that a rate of one event per ten years of observations within a redshift of $z < 0.5$ is not unrealistic.

3 SIGNATURE OF BURSTS WITH MEMORY IN PULSAR TIMING RESIDUAL

3.1 Pulsar timing residual due to a BWM

A gravitational wave propagating between the pulsar and observer leads to a modulation in the frequency of the observed pulsar signal given by (Estabrook & Wahlquist 1975; Sazhin 1978; Detweiler 1979)

$$\frac{\Delta\nu}{\nu_0} = \frac{1}{2c}\int_0^D d\lambda \left(e^i e^j \frac{\partial h_{ij}}{\partial t}\right)\Big|_{\rm path}, \tag{7}$$

where ν_0 is the unperturbed pulsar frequency in the absence of gravitational waves and $\Delta\nu(t) = \nu(t) - \nu_0$ is the variation of the observed pulsar frequency due to the presence of a gravitational wave characterized by the field h_{ij}. D is the distance from the pulsar to the observer and c is the speed of light (below we set $c = 1$). The expression in the brackets is evaluated along the light ray path from the pulsar to the observer, with the integration variable λ being the distance parameter along this path, and e^i being the spatial unit vector along the path. The unperturbed light ray path is given by

$$t(\lambda) = t - \lambda, \quad x^i(s) = x^i_O - e^i\lambda, \tag{8}$$

where t is the time of observation and x^i_O is the position of the observer. Without loss of generality we set $x^i_O = 0$ by choosing a spatial coordinate system with the observer at origin.

The pulsar timing measurements primarily measure the timing residuals, i.e. the difference between the actual pulse arrival times and times predicted from a spin-down model for a pulsar (Detweiler 1979). The timing residual $s(t)$, accumulated during a time interval of length t beginning from an initial time $t_{\rm in} = 0$, due to the presence of a gravitational wave can be calculated from the expression for the frequency modulation (7) in the following way

$$s\,(t) = \int_0^t d\tau \frac{\Delta\nu(\tau)}{\nu_0}. \tag{9}$$

the residual $s\,(t)$ has the dimensions of time and is customarily measured in nanoseconds.

In our analysis we shall assume that the gravitational wave source is sufficiently far from the observer on Earth in comparison to the distance D between the observer and the pulsar, so as to treat the GW in the plane wave approximation. In this case, the GW incoming from the direction given by the unit vector n_i can be presented in the form

$$h_{ij}\left(x^i, t\right) = h_+\left(t - n_i x^i\right) p^+_{ij} + h_\times\left(t - n_i x^i\right) p^\times_{ij}, \tag{10}$$

where h_+ and $h_\times$ are the amplitudes corresponding to two linear polarization states of a GW. The linear polarization tensors p^+_{ij} and $p^\times_{ij}$ can be expressed in terms of two mutually orthogonal unit vectors l_i and m_i lying perpendicular to the direction of the wave propagation n_i as follows $p^+_{ij} = (l_i l_j - m_i m_j)$ and $p^\times_{ij} = (l_i m_j + m_i l_j)$. The various projection terms that are encountered in evaluating expressions (7) and (9) can be written as

$$n_i e^i = \mu, \quad p^+_{ij} e^i e^j = (1-\mu^2)\cos 2\phi, \quad p^\times_{ij} e^i e^j = (1-\mu^2)\sin 2\phi, \tag{11}$$

in terms of angular variables $\mu = \cos\theta$ and ϕ (see, e.g., Baskaran et al. (2008)). Here θ is the angle between the direction of GW propagation n^i and the direction from the pulsar to the observer e^i. Angle ϕ is the azimuthal angle of e^i with respect to the principal direction l_i and m_i characterizing the GW polarization tensor, i.e. the angle between vector l_i and the projection of vector e^i onto $l_i m_i$-plane.

The expression for frequency modulation (7) can be integrated for the GW given in the form (10) exactly. Taking into account the projection factors (11), we get

$$\begin{aligned}\frac{\Delta\nu(t)}{\nu_0} = \quad & \frac{1}{2}\,(1+\mu)\,\Big\{\Big[h_+\,(t)\cos 2\phi + h_\times\,(t)\sin 2\phi\,\Big] - \\ & - \Big[h_+\,(t - D\,(1-\mu))\cos 2\phi + h_\times\,(t - D\,(1-\mu))\sin 2\phi\,\Big]\Big\}.\end{aligned} \tag{12}$$

The above expression has a clear physical interpretation, according to which the variation in frequency is directly proportional to the difference between the GW field strength at the place and time of observation (terms in the first square bracket) and its strength at the place and time of signal emission (terms in the second square brackets). In specific case of a BWM signal that reached the Earth during time span of pulsar observation, the second term will naturally vanish since the strength of the signal at the place and time of emission would have been equal to zero. On the other hand, it can be seen from (9) and (12) that if BWM front crossed the pulsar neighborhood at or before the moment of emission the second term would give rise to

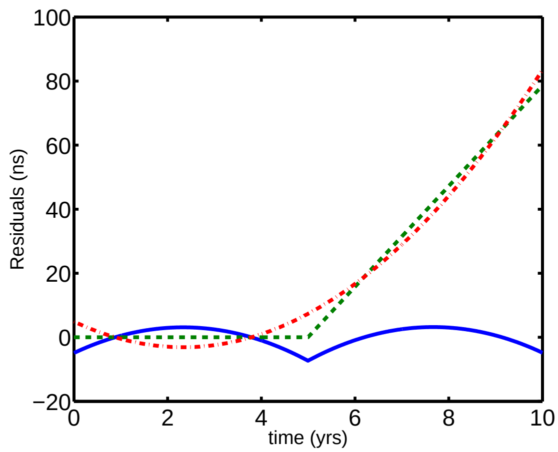

Figure 1. The solid line shows the expected timing residual due to a BWM with $h^{\rm mem} = 5 \cdot 10^{-16}$ and $t_B = 5$ yrs, in a pulsar with directional angles $\mu = \phi = 0$, during an observation spanning 10 yrs, after subtraction of the quadratic fitting term. The dashed and the dashed-dotted lines show the BWM residual signal before subtraction and quadratic subtraction terms, respectively.

linear rising term in prefit timing residuals. This linear term would be absorbed after fitting procedure (see below) and so the second term can be neglected in this case also. For these reasons, below we shall omit the second term in (12).

Using expression (12) and neglecting the terms proportional to the GW field strength at emission, we arrive at the expression for the timing residual

$$s(t) = \frac{1}{2}\,(1+\mu)\,\left\{\left(\int\limits_0^t d\tau\; h_+(\tau)\right)\cos 2\phi + \left(\int\limits_0^t d\tau\; h_\times(\tau)\right)\sin 2\phi\right\}. \tag{13}$$

We shall model the BWM in a simple analytical manner (that would be applicable in a wide range of situations) as a step-function signal

$$h_+(t) = h^{mem}\Theta(t-t_B), \quad h_\times(t) = 0, \tag{14}$$

where t_B is the time the BWM signal reaches the observer on Earth. The function $\Theta(x)$ is the Heaviside step function

$$\Theta(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases} \tag{15}$$

From (13) and (14) we find the prefit timing residuals:

$$s\left(t\right)_{\rm prefit} = \frac{1}{2}h^{\rm mem}(1+\mu)\cos 2\phi\;\left(t-t_{\rm B}\right)\Theta\left(t-t_{\rm B}\right), \tag{16}$$

Postfit residuals are obtained by the removal of linear and quadratic trends from prefit ones to take into account a priori unknown values of the pulsar period and its first derivative.

$$s\left(t\right) = \frac{1}{2}h^{\rm mem}(1+\mu)\cos 2\phi\;\mathcal{I}(t), \tag{17}$$

where $\mathcal{I}\left(t\right)$ is given by the expression

$$\mathcal{I}\left(t\right) = \left(t-t_{\rm B}\right)\Theta\left(t-t_{\rm B}\right) - \mathcal{I}_{\rm quad}(t). \tag{18}$$

The quadratic subtraction term $\mathcal{I}_{\rm quad}(t)$ can be written in a general form

$$\mathcal{I}_{\rm quad}\left(t\right) = a\left(t-t_{\rm B}\right)^2 + b\left(t-t_{\rm B}\right) + c. \tag{19}$$

The coefficients a, b and c can be evaluated by minimizing the integral $\int_0^{T_{\rm obs}} dt\; \mathcal{I}^2\left(t\right)$ (where $T_{\rm obs}$ is the total duration of observation) with respect to these variables. For example, in the case of a BWM signal occurring at $t_B = T_{\rm obs}/2$, these coefficients take a particularly simple form

$$a = \frac{15}{16T_{\rm obs}}, \quad b = \frac{1}{2}, \quad c = \frac{3T_{\rm obs}}{64}. \tag{20}$$

The signature in the pulsar timing residuals of such model BWM is shown in Fig. 1.

In the following subsections, we shall assess the sensitivity of PTA observations to the BWM signal, and estimate the expected rate of observable events.

3.2 Signal-to-noise ratio

In order to quantify the detection ability of PTA, in the present section we introduce and evaluate the signal-to-noise ratio for a BWM event. The observed residuals in a pulsar timing array can be presented in the form

$$R_\alpha(t_i) = s_\alpha(t_i) + n_\alpha(t_i)\,, \tag{21}$$

where the index $\alpha = 1, .., N_\alpha$ marks the residuals measured for the $\alpha^{\rm th}$ pulsar, and the index $i = 1, .., N_t$ is the number of the measurement. N_α and N_t are the number of pulsars in the timing array and the number of time observations per individual pulsar, respectively. $s_\alpha(t_i)$ is the part of residuals due to BWM (Eq. (17)), and $n_\alpha(t_i)$ is the noise.

We shall assume that $n_\alpha(t_i)$ is gaussian stationary white noise, uncorrelated for different pulsars. Under these assumptions, the noise correlation function will have the form

$$\overline{n_\alpha(t_i)\, n_\beta(t_j)} = \sigma_n^2(\alpha)\delta_{ij}\delta_{\alpha\beta}, \tag{22}$$

where $\delta_{\alpha\beta}$ and δ_{ij} are the Kronecker deltas. The residuals $s_\alpha(t)$ can be conveniently rewritten as

$$s_\alpha(t_i) = h^{\rm mem} f(\mu_\alpha, \phi_\alpha)\, \mathcal{I}(t_i)\,, \tag{23}$$

where $\mathcal{I}(t_i)$ is the common part of the signal for all pulsars (18), and $f(\mu_\alpha, \phi_\alpha)$ is the part of the residuals that depends on the orientation angles of the pulsars $(\mu_\alpha, \phi_\alpha)$

$$f(\mu_\alpha, \phi_\alpha) = \frac{1}{2}(1+\mu_\alpha)\cos 2\phi_\alpha. \tag{24}$$

The common way to extract a signal with a known form from a gaussian stationary noise is by using a matched filter (Sathyaprakash & Schutz 2009). The signal-to-noise ratio ρ attainable using the matched filter can be written in terms of the power of the noise $\sigma_n^2(\alpha)$ and the expected signal $s_\alpha(t_i)$

$$\rho^2 = \sum_{\alpha=1}^{N_\alpha} \left(\frac{1}{\sigma_n^2(\alpha)} \sum_{i=1}^{N_t} s_\alpha^2(t_i) \right). \tag{25}$$

Note that the above expression for signal-to-noise ratio is derived in time-domain variables which is more convenient in our case. Using (23), ρ can be rewritten in a factorized form

$$\rho^2 = \left(h^{\rm mem}\right)^2 N_t N_\alpha \left(\frac{1}{N_\alpha} \sum_{\alpha=1}^{N_\alpha} \frac{f^2(\mu_\alpha, \phi_\alpha)}{\sigma_n^2(\alpha)} \right) \left(\frac{1}{N_t} \sum_{i=1}^{N_t} \mathcal{I}^2(t_i) \right). \tag{26}$$

Assuming a near isotropic sky coverage for the pulsar timing array and a similar noise level for all the pulsars σ_n, the terms in the first bracket can be calculated as a sky-average value

$$\frac{1}{N_\alpha} \sum_{\alpha=1}^{N_\alpha} \frac{f^2(\mu_\alpha, \phi_\alpha)}{\sigma_n^2(\alpha)} \approx \frac{1}{4\pi\sigma_n^2} \int d\mu d\phi\; f^2(\mu, \phi) = \frac{1}{6\sigma_n^2}. \tag{27}$$

The terms in the second bracket in expression (26) can be approximated by an integral

$$\frac{1}{N_t} \sum_{i=1}^{N_t} \mathcal{I}^2(t_i) \approx \frac{1}{T_{\rm obs}} \int_0^{T_{\rm obs}} dt\; \mathcal{I}^2(t). \tag{28}$$

This integral can be calculated explicitly if the form of $\mathcal{I}_{\rm quad}$ in Eq. (18) is known. For example, in the case of $t_B = T_{\rm obs}/2$, the form of $\mathcal{I}_{\rm quad}$ is given by expressions (19) and (20) and

$$\int_0^{T_{\rm obs}} dt\; \mathcal{I}^2(t) = \frac{T_{\rm obs}^3}{3072}, \tag{29}$$

It is convenient to introduce a dimensionless quantity ι

$$\iota = \sqrt{\frac{\int_0^{T_{\rm obs}} dt\; \mathcal{I}^2(t)}{T_{\rm obs}^3/3072}}, \tag{30}$$

which is about one for $0 < t_{\rm B} < T_{\rm obs}$.

In the future PTA observations $N_t = 250$, $N_\alpha = 20$, $T_{\rm obs} = 10$ yrs, $\sigma_n = 100$ ns (that is the level of sensitivity currently achieved for a number of pulsars (Verbiest et al. 2009)). Taking these numbers along with a signal strength $h^{\rm mem} = 10^{-15}$ as guidelines and using approximations (27) and (28), the expression for signal-to-noise ratio can be rewritten in the form

$$\rho = 1.64 \left[\iota \left(\frac{h^{\rm mem}}{10^{-15}}\right) \left(\frac{N_t}{250}\right)^{\frac{1}{2}} \left(\frac{N_\alpha}{20}\right)^{\frac{1}{2}} \left(\frac{T_{\rm obs}}{10{\rm yrs}}\right) \left(\frac{100{\rm ns}}{\sigma_n}\right) \right]. \tag{31}$$

3.3 Expected event rate

Equation (31) shows the practical possibility of using pulsar timing measurements to detect individual GW bursts that accompany SMBH mergers. As can be seen from Eq. (31), the BWM with an amplitude of $h^{\rm mem} \sim (1.5-2)\cdot 10^{-15}$ can be detected in a 10-years run of PTA observations with a signal-to-noise ratio of ~ 3. Eq. (5) indicates that such a signal would be produced by the coalescence of two SMBH with equal masses around $3.5\cdot 10^8\ M_\odot$ at a distance of 1 Gpc.

It is instructive to substantiate this rate by properly taking into account the contribution to event rate from SMBH mergers at various cosmological distances (redshifts). The rate of GW bursts of sufficient strength from coalescing SMBH coming from redshifts $z < z_{\rm lim}$, $\dot{N}(z_{\rm lim})$, ($[\dot{N}(z_{\rm lim})] = {\rm yr}^{-1}$) can be calculated from the number density of SMBH mergers $n(z)$ with masses $M_{\rm BH} > M_{\rm lim}(z)$, ($[n(z)] = {\rm Mpc}^{-3}$) and the characteristic merging frequency for a SMBH, $\eta(z)$, $[\eta] = {\rm yr}^{-1}$. The event rate $\dot{N}(z_{\rm lim})$ of bursts with sufficient strength is

$$\dot{N}(z_{\rm lim}) = \int_0^{z_{\rm lim}} n_0(1+z)^3\frac{\eta(z)}{1+z}4\pi r^2 dr\,, \tag{32}$$

where r is the metric distance determined from

$$\frac{dr}{dz} = \frac{c}{H_0}\frac{1}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}} \tag{33}$$

in a flat ΛCDM cosmological model. Here H_0 is the present-day value of the Hubble parameter and Ω_Λ, Ω_m stand for the cosmological constant and matter energy content in units of the critical density, $\Omega_m + \Omega_\Lambda = 1$. For numerical estimations below we adopt the standard values $H_0 = 72$ km s^{-1}Mpc^{-1}, $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$.

The local comoving density of SMBH n_0 can be estimated using the integral SMBH mass-function from the study of Caramete & Biermann (2009):

$$\phi(M_{\rm BH}) \equiv N(M > M_{\rm BH}) = 7\cdot 10^{-2}\left(\frac{M_{\rm BH}}{10^7\ M_\odot}\right)^{-2}\ {\rm Mpc}^{-3}\,. \tag{34}$$

The local SMBH merging rate η_0 can be estimated, assuming that each SMBH with the mass greater than 10^8 M$_\odot$ has undergone at least one merging in the past, from the local density of SMBH n_0 and the specific merging rate per unit volume $\mathcal{R}_0$. Using the common assumption (e.g. Enoki et al. (2004)) that each major galaxy merging is associated with a SMBH coalescence, the specific SMBH merging rate can be obtained from the analysis of major mergings of galaxies with stellar mass $M_* > 10^{10}$ M$_\odot$ (Conselice et al. 2009), $\mathcal{R}_0 = 10^6$ Gpc^{-3}Gyr^{-1}.

$$\eta_0 = \frac{\mathcal{R}_0}{\rho_0} \approx \frac{10^{-12}\ {\rm Mpc}^{-3}{\rm yr}^{-1}}{10^{-2}\ {\rm Mpc}^{-3}} = 10^{-10}\ {\rm yr}^{-1}\,. \tag{35}$$

In this estimation we have used the local density of massive galaxies $\rho_0 \sim 0.01$ Mpc^{-3}. The obtained value agrees with the frequency of major galaxy mergers from (Conselice et al. 2009) $\eta_0 = (1-2)\cdot 10^{-10}$ yr^{-1}.

We are interested only in the SMBH merger events that could be registered with PTA. For this reason, more distant SMBH mergings should be more massive to be detected, and the corresponding minimal detectable mass scales as

$$M_{\rm lim}(z) \propto r(1+z)\,.$$

Setting the reference point at $r_0 = 1$ Gpc ($z \sim 0.2$) we get the corresponding reference values for SMBH mass M_0 and local comoving density of SMBH n_0:

$$M_0 = 3.5\cdot 10^8\ M_\odot,\quad n_0 = 6\cdot 10^{-5}\ {\rm Mpc}^{-3}. \tag{36}$$

In terms of the reference values, the minimum detectable mass can now be expressed in the form:

$$M_{\rm lim}(z) = M_0\left(\frac{r(1+z)}{r_0(1+z_0)}\right). \tag{37}$$

Using Eqs. (34), (36) and (37), we arrive at

$$n(z) = n_0\left(\frac{M_0}{M_{\rm lim}(z)}\right)^2 = 6\cdot 10^{-5}\left(\frac{r_0(1+z_0)}{r(1+z)}\right)^2 \approx 10^{-4}\left(\frac{r_0}{r(1+z)}\right)^2. \tag{38}$$

Substituting Eq. (38) into Eq. (32) we find

$$\dot{N}(z_{\rm lim}) = 4\pi\cdot 10^{-14} r_0^2\int_0^{z_{\rm lim}} dr = 4\pi\cdot 10^{-14} r_0^2\int_0^{z_{\rm lim}}\frac{dr}{dz}dz \tag{39}$$

After substituting (33) into above equation, we finally obtain

$$\begin{aligned}\dot{N}(z_{\rm lim}) &= 4\pi\cdot 10^{-14}\frac{r_0^2 c}{H_0}\int_0^{z_{\rm lim}}\frac{1}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}}dz \\ &= 5\cdot 10^{-4}\int_0^{z_{\rm lim}}\frac{1}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}}dz.\end{aligned} \tag{40}$$

We set our limiting redshift to $z_{\rm lim} = 5$, bearing in mind that SMBH assembly increasingly occurs around $z \sim 3$ (see, e.g., Enoki et al. (2004)). In that case the value of the integral in Eq. (40) is close to 2 and the detection rate is

$$\dot{N}(z_{\rm lim} = 5) \approx 10^{-3}\ {\rm yr}^{-1}\,. \tag{41}$$

Taken at face value, this rate seems to be fairly low, but we can increase it significantly if we take into account the strong redshift dependence of the specific major galaxy merging rate $\eta(z) = \eta_0(1+z)^\beta$. With this factor the integral in Eq. (40) takes the approximate value:

$$\int_0^5 \frac{1}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}}(1+z)^\beta dz \approx 1.25\cdot 10^{0.58\beta}\,. \tag{42}$$

This approximation slightly underestimates the integral for $\beta \sim 0$. The analysis of (Conselice et al. 2009) suggests $\beta = 2-3$. Adopting $\beta = 2$ we obtain the SMBH detection rate from the limiting redshift $z_{\rm lim} = 5$

$$\dot{N}(z_{\rm lim} = 5) \approx 6\cdot 10^{-4+0.58\beta}\ {\rm yr}^{-1} \approx 10^{-2}\ {\rm yr}^{-1}\,. \tag{43}$$

Our figure of merit is the number of detections with SNR above a fixed threshold value ρ in a PTA observation with given sensitivity (i.e. the rms of timing residuals σ_n) over the total observing time $T_{\rm obs}$. This number is determined by the product $\dot{N}\times T_{\rm obs}$. The detection rate $\dot{N}\,(z_{\rm lim}) \propto n_0(M_0) \propto M_0^{-2} \propto (h^{\rm mem})^{-2}$, where $h^{\rm mem}$ is the signal amplitude that triggers the detector at a given SNR level ρ (see Eq. (31)), i.e. $h^{\rm mem} \propto \rho N_t^{-1/2} N_\alpha^{-1/2} T_{\rm obs}^{-1}\sigma_n$. So finally, we find the number of detections in observations with duration $T_{\rm obs}$ is

$$N \simeq 10^{-1}\left(\frac{N_t}{250}\right)\left(\frac{N_\alpha}{20}\right)\left(\frac{T_{\rm obs}}{10\ {\rm yrs}}\right)^3\left(\frac{100\ {\rm ns}}{\sigma_n}\right)^2\left(\frac{3}{\rho}\right)^2\,. \tag{44}$$

It can be seen that the expected number of detections is very sensitive to the duration of observations and the rms noise level.

4 CONCLUSIONS

We have shown that future pulsar timing measurements will be capable of detecting individual gravitational wave bursts that should accompany SMBH mergers. A GW burst with memory with an amplitude of $\sim (1.5-2)\cdot 10^{-15}$ leaves the characteristic imprint in the pulsar timing residuals and can be detected in a 10-years run of Pulsar Timing Array observations with the present-day characteristics at a signal-to-noise ratio of ~ 3. Such a signal is expected to be produced by the coalescence of two SMBH with equal masses around $3.5\cdot 10^8\ M_\odot$ at a distance of 1 Gpc. We estimate the rate of SMBH coalescences producing the BWM with such an amplitude to be around a few hundredth per year. The number of detections of such GW bursts from SMBH mergings by a PTA array with given characteristics is given by Eq. (44) that shows a strong dependence on the total time of observations and the noise level in pulsar timing residuals. It is expected that increase in the PTA sensitivity in the near future will allow detection of BWM from coalescing SMBHs at a signal-to-noise ratio more than three. Future radiotelescopes, especially SKA, will be able to increase the sensitivity several times due to the decrease in timing residual noise and the extension of number of pulsars in pulsar timing program, thus making this method complementary to the LISA space mission for detection of coalescing SMBHs with masses $> 10^8\ M_\odot$ to which the sensitivity of LISA is reduced.

Finally, it is worth pointing out that the analysis conducted in this paper is applicable for any source of BWM, not specifically restricted to GW signals from SMBH mergers. As was mentioned above, BWM are a generic feature of GW sources that emit energy in an asymmetric fashion. These BWM events will leave an imprint in pulsar timing residuals as long as they are sufficiently bright $h^{\rm mem} \gtrsim 10^{-15}$.

When this paper was already submitted, two studies dealing with the influence of BWM on PTA appeared (Seto 2009; van Haasteren & Levin 2009). Results of (van Haasteren & Levin 2009) are based on Bayesian analysis method and give the same conclusions, results of (Seto 2009) with different treatment of background noise is somewhere more pessimistic.

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